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MEAN AND EDDY MOTIONS IN THE ATMOSPHERE [1]

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ABSTRACT

A review is given of mathematical expressions for total and mean specific kinetic energies in the longitudinal, time, and mixed longitudinal-time domains. These coordinate domains differ from those defined earlier by Oort. Mathematical developments are extended into the vertical coordinate domain. A new symbolism is introduced for describing mean and eddy motions.

1. INTRODUCTION

In recent years a large number of studies—too numerous to be listed here—were concerned with energy conversions, energy and mass transports, and the transports of trace substances in the atmosphere. Invariably these studies consider "mean" and "eddy" fluxes of atmospheric properties, the latter being defined as the departures from the former, and the sum of both describing the total flux. As long as only time or space averages are dealt with, mean and eddy quantities are easily defined and calculated. If, however, both time and longitudinal "domains" are regarded simultaneously, matters become fairly complicated, as has been pointed out by Oort (1964). Vertical averages, and departures thereof, would yield the barotropic and baroclinic contributions to energy conversion and transport processes. These have been considered separately by Wiin-Nielsen and Drake (1965, 1966). If one should wish to consider the vertical coordinate together with the horizontal space and the time coordinates, the mathematical problem with present symbolism becomes unwieldy. This paper introduces a new symbolic language for mean and eddy values, which is more lengthy than the one introduced by Osborne Reynolds and subsequently adopted—with modifications—by many authors. It has the advantage, however, that in each term the sequence of averaging and departure-forming steps may be viewed easily, even if several coordinates are involved in the averaging process.

In an earlier study, Lorenz (1953) proposed a symbolism based upon the use of multiple subscripts 0, 1, and 2 and the permutations thereof which, however, he did not use in later publications. Although the "philosophy" of the notation presented here is rather similar to the one by Lorenz, it has the following additional advantages:

1) The use of subscripts is not preempted for standard mathematical notations (e.g., indicating derivatives or vector components).

2) The present notation allows a clear and unambiguous indication of the sequence in which multiplication steps (for instance in forming products or squares of certain values) and averaging or departure-forming steps have to be taken.

3) The derivations presented here also include products of linear perturbation terms which may bear out certain correlations. They should not be neglected before their magnitudes have been estimated.

4) The notation presented here is easier to typeset than the one using asterisks, primes, bars, tildes, etc.

2. DEFINITION OF SYMBOLS

Averaging processes will be indicated by brackets, departures from these averages by parentheses. Subscripts in parentheses give the coordinates over which mean values and departures thereof, were computed. Thus,

$$u=[u]_{(t)}+(u)_{(t)} \tag{1}$$

stands for the familiar sum of time average and fluctuations about this average, which yields the instantaneous values of u. It follows from this definition that

$$[[u]_{(t)}]_{(t)}=[u]_{(t)} \tag{2}$$

and

$$[(u)_{(t)}]_{(t)}=0,$$

$$[([(u)_{(t)}]_{(i)})_{(j)}]_{(t)}=0; \tag{3}$$

also,

$$([u]_{(t)})_{(t)}=0,$$

$$([([u]_{(t)})_{(i)}]_{(j)})_{(t)}=0, \tag{4}$$

i and j being coordinates different from t.

For an additional averaging process, for example with respect to geographic longitude, the conditions (1) to (4) hold, as well as

$$[u]_{(\lambda,\, t)}=[u]_{(t,\,\lambda)} \tag{5}$$

if the sequence of the subscripts is to indicate the sequence in which the averaging processes are performed. Although condition (5) is mathematically true for distributions of u, which are steady in both time and space, there will be a difference between the left and right side of (5) in actual computations. These discrepancies are caused by differences in measurement and computational errors, when

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evaluating the distribution of u first in a space grid and then computing the time average for each grid point, or when computing time-mean values of u for each observation station, and subsequently interpolating gridpoint values, averaging the latter with respect to longitude.

In the subsequent discussion we will assume—for the sake of simplicity—that condition (5) is true. It should be mentioned, however, that an inequality in (5) will yield valuable estimates on the accuracy of subjective or objective techniques by which gridpoint values of u were obtained.

3. LONGITUDINAL AND TIME COORDINATES

From equation (1) one may derive double the value of the zonal component of specific kinetic energy (i.e., energy per unit mass)

$$u^2[=u]_{(t)}\cdot[u]_{(t)}+2[u]_{(t)}\cdot(u)_{(t)}+(u)_{(t)}\cdot(u)_{(t)}$$

or

$$u^2=[u]^2_{(t)}+2[u]_{(t)}\cdot(u)_t+(u)^2_{(t)}. \tag{6}$$

On the right side of this equation we may substitute

$$u=[u]_{(\lambda)}+(u)_{(\lambda)}. \tag{7}$$

The following terms, which will result from this substitution, are listed in the sequence of their appearance in equation (6):

$$\begin{aligned}u^2=&\overset{\textcircled{1}}{[u]^2_{(\lambda,t)}}+\overset{\textcircled{2}}{2[u]_{(\lambda,t)}\cdot[(u)_{(\lambda)}]_{(t)}}+\overset{\textcircled{3}}{[(u)_{(\lambda)}]^2_{(t)}}\\&+\overset{\textcircled{4}}{2[u]_{(\lambda,t)}\cdot([u]_{(\lambda)})_{(t)}}+\overset{\textcircled{5}}{2[u]_{(\lambda,t)}(u)_{(\lambda,t)}}\\&+\overset{\textcircled{6}}{2[(u)_{(\lambda)}]_{(t)}([u]_{(\lambda)})_{(t)}}+\overset{\textcircled{7}}{2[(u)_{(\lambda)}]_{(t)}(u)_{(\lambda,t)}}\\&+\overset{\textcircled{8}}{([u]_{(\lambda)})^2_{(t)}}+\overset{\textcircled{9}}{2([u]_{(\lambda)})_{(t)}\cdot(u)_{(\lambda,t)}}\\&+\overset{\textcircled{10}}{(u)^2_{(\lambda,t)}}.\end{aligned} \tag{8}$$

Terms on the right side of this equation have been numbered for convenient reference. A similar expression may be derived for v^2. The instantaneous and local total specific kinetic energy will then be given by

$$K=\tfrac{1}{2}(u^2+v^2). \tag{9}$$

We will now define the "space domain" or, more appropriately, the "longitudinal domain." The mean kinetic energy in this domain is given by $[K]_{(\lambda)}$ and differs from Oort's definition. Double the contribution towards this quantity by the x-component of motion is obtained by averaging equation (8) with respect to λ.

$$\begin{aligned}[u^2]_{(\lambda)}=&\overset{\textcircled{1}}{[u]^2_{(\lambda,t)}}+\overset{\textcircled{3}}{[[(u)_{(\lambda)}]^2_{(t)}]_{(\lambda)}}+\overset{\textcircled{4}}{2[u]_{(\lambda,t)}\cdot([u]_{(\lambda)})_{(t)}}\\&+\overset{\textcircled{7}}{2[[(u)_{(\lambda)}]_{(t)}\cdot(u)_{(\lambda,t)}]_{(\lambda)}}+\overset{\textcircled{8}}{([u]_{(\lambda)})^2_{(t)}}\\&+\overset{\textcircled{10}}{[(u)^2_{(\lambda,t)}]_{(\lambda)}}.\end{aligned} \tag{10}$$

(Terms in this equation are numbered in correspondence with equation (8).) All other terms will vanish because they contain averaging processes of the form $[(u)_{(\lambda)}]_{(\lambda)}$ which, according to condition (3), will be zero. Term ⑦ has been retained because of a possible correlation between $[(u)_{(\lambda)}]_{(t)}$ and $(u)_{(\lambda,t)}$.

Term ① stands for the kinetic energy of the mean motion, term ③ contributes towards the kinetic energy of *standing eddies* (*nota bene* $[(u)^2_{(\lambda)}]_{(\lambda)}\neq 0$), and terms ⑧ and ⑩ towards the effect of *transient eddies*. Term ④ indicates the "kinetic energy of pulsation," i.e., the zonally averaged mean zonal transport by *time* pulsations of the mean zonal momentum. This term, as well as term ⑧, will vanish if equation (10) is averaged over both λ and t.

The *eddy* kinetic energy in the longitudinal domain may be defined as $K-[K]_{(\lambda)}$. Double the contribution from the u-component of motion is obtained by subtracting equation (10) from equation (8).

The *mean kinetic energy* in the time domain may be computed in a similar fashion by considering

$$\begin{aligned}[u^2]_{(t)}=&\overset{\textcircled{1}}{[u]^2_{(\lambda,t)}}+\overset{\textcircled{2}}{2[u]_{(\lambda,t)}\cdot[(u)_{(\lambda)}]_{(t)}}+\overset{\textcircled{3}}{[(u)_{(\lambda)}]^2_{(t)}}\\&+\overset{\textcircled{8}}{[([u]_{(\lambda)})^2_{(t)}]_{(t)}}+\overset{\textcircled{9}}{2[([u]_{(\lambda)})_{(t)}\cdot(u)_{(\lambda,t)}]_{(t)}}\\&+\overset{\textcircled{10}}{[(u)^2_{(\lambda,t)}]_{(t)}}.\end{aligned} \tag{11}$$

All other terms, again, vanish for reasons stated before. Term ① gives a measure of the kinetic energy of the mean motion, term ③ of the energy of *standing eddies*, terms ⑧ and ⑩ of *transient eddies*, and term ② stands for the kinetic energy of pulsations produced by *local* variations of the zonal momentum. Term ⑨, again, has been retained because of a possible correlation between the two factors. Terms ② and ⑨, again, will vanish if equation (11) is averaged over both λ and t.

The *eddy kinetic energy* in the time domain is defined and may be computed by subtracting equation (11) from equation (8). Analogous computations will have to be carried out for the v-component of flow.

The mixed longitudinal and time domain has a *mean kinetic energy* given by $[K]_{(\lambda,t)}$. Its u-component contribution may be obtained from equation (8) by applying the operation $[\quad]_{(\lambda,t)}$:

$$\begin{aligned}[u^2]_{(\lambda,t)}=&\overset{\textcircled{1}}{[u]^2_{(\lambda,t)}}+\overset{\textcircled{3}}{[[(u)_{(\lambda)}]^2_{(t)}]_{(\lambda)}}+\overset{\textcircled{8}}{[([u]_{(\lambda)})^2_{(t)}]_{(t)}}\\&+\overset{\textcircled{10}}{[(u)^2_{(\lambda,t)}]_{(\lambda,t)}}.\end{aligned} \tag{12}$$

All linear terms containing parentheses, naturally, will vanish, including all "pulsation" terms. Term ③ contains standing eddy effects, terms ⑧ and ⑩ *transient eddy* effects.

It is of interest to note that the longitudinal domain and the time domain, as well as the mixed longitudinal-time domain, all contain effects of standing and transient eddies, averaged in different but specific ways.

The *eddy kinetic energy* in the mixed longitude-time domain is given by $K-[K]_{(\lambda,t)}=(K)_{(\lambda,t)}$.

The approach taken by Oort (1964) in defining the various domains is slightly different. (For the sake of brevity we will omit the v-component as well as the integration $\frac{1}{2}\int dm$ over the vertical extent of the atmosphere, both contained in Oort's notation, from our consideration.) In his definition the values of $[K]_{(\lambda,t)}$ given by equation (12) are decomposed into eddy and mean contributions. Thus, $[K]_{(\lambda,t)}$ is equivalent to the "total kinetic energy" considered by Oort.

Oort's definition of the space domain contains the terms of equation (12) with $[\ \ ]_{(\lambda)}$ as *mean* contribution (i.e., (1) and (8)), and the terms with $(\ \ )_{(\lambda)}$ as *eddy* contribution (i.e., (3) and (10)). Oort's time domain consists of the mean terms $[\ \ ]_{(t)}$ (i.e., (1) and (3)), and of the eddy terms $(\ \ )_{(t)}$ (i.e., (8) and (10)). Oort's mixed domain has only term (1) of equation (12) with $[\ \ ]_{(\lambda,t)}$ as mean contribution. All other terms containing at least one set of parentheses contribute towards the eddy energy.

4. THE VERTICAL COORDINATE

Following a suggestion by Wiin-Nielsen and Drake (1965), we may define a vertical averaging process along the coordinate p by

$$u=[u]_{(p)}+(u)_{(p)}. \tag{13}$$

$[u]_{(p)}$ symbolizes the barotropic contribution towards the atmospheric (zonal) flow, $(u)_{(p)}$ the baroclinic contribution. Substitution of this expression into equation (8) yields the following equation (14). As will be shown later, many of the terms contained in equation (14) will vanish with suitable averaging procedures. Nevertheless, they have been retained here for the sake of completeness. Computation of eddy transport processes within certain longitude sectors and during time periods $\tau<t$ might also make it mandatory to retain the complete set of terms.

Term number in equation (8) — Equation (14)

(1)
$$u^2=\overset{(1)}{[u]^2_{(p,\lambda,t)}}+\overset{(2)}{2[u]_{(p,\lambda,t)}\cdot[(u)_{(p)}]_{(\lambda,t)}}+\overset{(3)}{[(u)_{(p)}]^2_{(\lambda,t)}}$$

(2)
$$\begin{cases}+\overset{(4)}{2[u]_{(p,\lambda,t)}\cdot[([u]_{(p)})_{(\lambda)}]_{(t)}}+\overset{(5)}{2[u]_{(p,\lambda,t)}\cdot[(u)_{(p,\lambda)}]_{(t)}}\\ +\overset{(6)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot[([u]_{(p)})_{(\lambda)}]_{(t)}}\\ +\overset{(7)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot[(u)_{(p,\lambda)}]_{(t)}}\end{cases}$$

(3)
$$\begin{cases}+\overset{(8)}{[([u]_{(p)})_{(\lambda)}]^2_{(t)}}+\overset{(9)}{2[([u]_{(p)})_{(\lambda)}]_{(t)}\cdot[(u)_{(p,\lambda)}]_{(t)}}\\ +\overset{(10)}{[(u)_{(p,\lambda)}]^2_{(t)}}\end{cases}$$

(4)
$$\begin{cases}+\overset{(11)}{2[u]_{(p,\lambda,t)}\cdot([u]_{(p,\lambda)})_{(t)}}+\overset{(12)}{2[u]_{(p,\lambda,t)}\cdot([(u)_{(p)}]_{(\lambda)})_{(t)}}\\ +\overset{(13)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot([u]_{(p,\lambda)})_{(t)}}\\ +\overset{(14)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot([(u)_{(p)}]_{(\lambda)})_{(t)}}\end{cases}$$

(5)
$$\begin{cases}+\overset{(15)}{2[u]_{(p,\lambda,t)}\cdot([u]_{(p)})_{(\lambda,t)}}+\overset{(16)}{2[u]_{(p,\lambda,t)}\cdot(u)_{(p,\lambda,t)}}\\ +\overset{(17)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot([u]_{(p)})_{(\lambda,t)}}+\overset{(18)}{2[(u)_{(p)}]_{(\lambda,t)}\cdot(u)_{(p,\lambda,t)}}\end{cases}$$

(6)
$$\begin{cases}+\overset{(19)}{2[([u]_{(p)})_{(\lambda)}]_{(t)}\cdot([u]_{(p,\lambda)})_{(t)}}\\ +\overset{(20)}{2[([u]_{(p)})_{(\lambda)}]_{(t)}\cdot([(u)_{(p)}]_{(\lambda)})_{(t)}}\\ +\overset{(21)}{2[(u)_{(p,\lambda)}]_{(t)}\cdot([u]_{(p,\lambda)})_{(t)}}\\ +\overset{(22)}{2[(u)_{(p,\lambda)}]_{(t)}\cdot([(u)_{(p)}]_{(\lambda)})_{(t)}}\end{cases}$$

(7)
$$\begin{cases}+\overset{(23)}{2[([u]_{(p)})_{(\lambda)}]_{(t)}\cdot([u]_{(p)})_{(\lambda,t)}}\\ +\overset{(24)}{2[([u]_{(p)})_{(\lambda)}]_{(t)}\cdot(u)_{(p,\lambda,t)}}\\ +\overset{(25)}{2[(u)_{(p,\lambda)}]_{(t)}\cdot([u]_{(p)})_{(\lambda,t)}}+\overset{(26)}{2[(u)_{(p,\lambda)}]_{(t)}\cdot(u)_{(p,\lambda,t)}}\end{cases}$$

(8)
$$\begin{cases}+\overset{(27)}{([u]_{(p,\lambda)})^2_{(t)}}+\overset{(28)}{2([u]_{(p,\lambda)})_{(t)}\cdot([(u)_{(p)}]_{(\lambda)})_{(t)}}\\ +\overset{(29)}{([(u)_{(p)}]_{(\lambda)})^2_{(t)}}\end{cases}$$

(9)
$$\begin{cases}+\overset{(30)}{2([u]_{(p,\lambda)})_{(t)}\cdot([u]_{(p)})_{(\lambda,t)}}+\overset{(31)}{2([u]_{(p,\lambda)})_{(t)}\cdot(u)_{(p,\lambda,t)}}\\ +\overset{(32)}{2([(u)_{(p)}]_{(\lambda)})_{(t)}\cdot([u]_{(p)})_{(\lambda,t)}}\\ +\overset{(33)}{2([(u)_{(p)}]_{(\lambda)})_{(t)}\cdot(u)_{(p,\lambda,t)}}\end{cases}$$

(10)
$$+\overset{(34)}{([u]_{(p)})^2_{(\lambda,t)}}+\overset{(35)}{2([u]_{(p)})_{(\lambda,t)}\cdot(u)_{(p,\lambda,t)}}+\overset{(36)}{(u)^2_{(p,\lambda,t)}}$$

In order to estimate the mean kinetic energies in the various domains we will, for the sake of brevity, indicate only the term numbers of equation (14), which enter into the expressions of kinetic energy. It should be kept in mind that these terms will have to be divided by two, and appropriate terms for the v-components will have to be added, in order to arrive at the specific kinetic energies. In the equations following below all terms have been retained which might give a correlation of the form $(u)_{(p)}\ (u)_{(p)}$, etc.

Mean kinetic energy in the longitude domain:

$$
\begin{array}{l}
[u^2]_{(\lambda)}=① \quad +② \quad +③ \quad \cdot \quad \cdot \quad \cdot \\
\quad +[⑧]_{(\lambda)}+[⑨]_{(\lambda)}+[⑩]_{(\lambda)}+ ⑪ \quad + ⑫ \\
\quad +⑬ \quad +⑭ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad +[㉓]_{(\lambda)}+[㉔]_{(\lambda)} \\
\quad +[㉕]_{(\lambda)}+[㉖]_{(\lambda)}+ ㉗ \quad + ㉘ \quad + ㉙ \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad +[㉞]_{(\lambda)}+[㉟]_{(\lambda)}+[㊱]_{(\lambda)}.
\end{array} \tag{15}
$$

Mean kinetic energy in the time domain:

$$
\begin{array}{l}
[u^2]_{(t)}=① \quad +② \quad +③ \quad +④ \quad +⑤ \quad +⑥ \\
\quad +⑦ \quad +⑧ \quad +⑨ \quad +⑩ \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad +[㉗]_{(t)}+[㉘]_{(t)}+[㉙]_{(t)}+[㉚]_{(t)} \\
\quad +[㉛]_{(t)}+[㉜]_{(t)}+[㉝]_{(t)}+[㉞]_{(t)}+[㉟]_{(t)}+[㊱]_{(t)}.
\end{array} \tag{16}
$$

Mean kinetic energy in the vertical domain:

$$
\begin{array}{l}
[u^2]_{(p)}=① \quad \cdot \quad +[③]_{(p)}+④ \quad \cdot \quad \cdot \\
\quad +[⑦]_{(p)}+⑧ \quad \cdot \quad +[⑩]_{(p)}+⑪ \quad \cdot \\
\quad \cdot \quad +[⑭]_{(p)}+⑮ \quad \cdot \quad \cdot \quad +[⑱]_{(p)} \\
\quad +⑲ \quad \cdot \quad \cdot \quad +[㉒]_{(p)}+㉓ \quad \cdot \\
\quad \cdot \quad +[㉖]_{(p)}+㉗ \quad \cdot \quad +[㉙]_{(p)}+㉚ \\
\quad \cdot \quad \cdot \quad +[㉝]_{(p)}+㉞ \quad \cdot \quad +[㊱]_{(p)}.
\end{array} \tag{17}
$$

Mean kinetic energy in the longitude-time domain:

$$
\begin{array}{l}
[u^2]_{(\lambda,t)}=①+② \quad +③ \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot +[⑧]_{(\lambda)}+[⑨]_{(\lambda)}+[⑩]_{(\lambda)} \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad +[㉗]_{(t)}+[㉘]_{(t)} \quad +[㉙]_{(t)} \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad +[㉞]_{(\lambda,t)}+[㉟]_{(\lambda,t)}+[㊱]_{(\lambda,t)}.
\end{array} \tag{18}
$$

Mean kinetic energy in the longitude-vertical (space) domain:

$$
\begin{array}{l}
[u^2]_{(p,\lambda)}=① \quad \cdot \quad +[③]_{(p)} \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot +[⑧]_{(\lambda)} \quad \cdot \quad +[⑩]_{(p,\lambda)}+⑪ \quad \cdot \\
\quad \cdot +[⑭]_{(p)} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad +[㉓]_{(\lambda)} \quad \cdot \\
\quad \cdot +[㉖]_{(p,\lambda)}+㉗ \quad \cdot \quad +[㉙]_{(p)} \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad +[㉞]_{(\lambda)} \quad \cdot \quad +[㊱]_{(p,\lambda)}.
\end{array} \tag{19}
$$

Mean kinetic energy in the vertical-time domain:

$$
\begin{array}{l}
[u^2]_{(p,t)}=① \quad \cdot +[③]_{(p)} \quad +④ \quad \cdot \quad \cdot \\
\quad +[⑦]_{(p)}+⑧ \quad \cdot \quad +[⑩]_{(p)} \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot +[㉗]_{(t)} \quad \cdot \quad +[㉙]_{(p,t)}+[㉚]_{(t)} \\
\quad \cdot \quad \cdot +[㉝]_{(p,t)}+[㉞]_{(t)} \quad \cdot \quad +[㊱]_{(p,t)}.
\end{array} \tag{20}
$$

Mean kinetic energy in the space-time domain:

$$
\begin{array}{l}
[u^2]_{(p,\lambda,t)}=① \quad \cdot \quad +[③]_{(p)} \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot +[⑧]_{(\lambda)} \quad \cdot \quad +[⑩]_{(p,\lambda)}] \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad +[㉗]_{(t)} \quad \cdot \quad +[㉙]_{(p,t)} \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad [㉞]_{(\lambda,t)} \quad \cdot \quad +[㊱]_{(p,\lambda,t)}.
\end{array} \tag{21}
$$

Expressions (15), (16), and (17), which represent the domains of a simple coordinate, each contain 20 terms. Those terms written without brackets will not be altered by the additional averaging processes. They may be taken as they appear in equation (14). Bracketed terms will have to be subjected to an averaging process along the coordinate(s) indicated in the subscript(s).

Expressions (18), (19), and (20) represent mixed domains of two simultaneous coordinates. They contain 12 terms each, namely those that appear simultaneously in the two expressions representing the appropriate two single-coordinate domains. Expression (21) stands for the mixed space-time domain (the "space" being assumed two-dimensional). Its eight terms are those that appear simultaneously in equations (15), (16), and (17). They are the *quadratic* terms of equation (14). All linear terms vanish in the additional triple-averaging processes.

The appropriate *eddy* kinetic energies may be found by subtracting the mean kinetic energies from the total kinetic energy $K=\frac{1}{2}(u^2+v^2)$, which may be obtained from equation (14) and from a similar equation for the v-component.

Following Oort's (1964) approach one might decompose expression (21) into mean and eddy terms, similar to the discussion subsequent to equation (12). Table 1 gives a schematic view of all possible decompositions of $[u^2]_{(p,\lambda,t)}$.

The significance of these terms can easily be interpreted. For example, standing eddies appear under the category "time mean" in table 1. Barotropic contributions are listed under "vertical mean." The barotropic contributions from standing eddies are found under vertical-time mean. Transient eddies and baroclinic contributions are listed in the appropriate categories of the eddy column.

5. CONCLUSIONS

For most practical purposes, when one is concerned only with the average characteristics of the atmosphere throughout its depth, on a global basis, and during a certain period of time (e.g., 1 mo), equation (21) will give all required detials. It describes standing, transient, barotropic, and baroclinic eddy effects, superimposed upon the mean motion (see table 1). Only eight terms out of the 36 listed in equation (14), therefore, will have to be retained. To the best of the author's knowledge, no attempts have been made yet to estimate the relative magnitudes and the importance of these terms describing

TABLE 1.—*Decomposition of* $[u^2]_{(p,\lambda,t)}$. *Term numbers refer to equation (14) and appear as listed in equation (21).*

Longitude mean	Longitude eddies
$1+[3]_{(p)}+[27]_{(t)}+[29]_{(p,t)}$	$[8]_{(\lambda)}+[10]_{(p,\lambda)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$
Time mean	Time eddies
$1+[3]_{(p)}+[8]_{(\lambda)}+[10]_{(p,\lambda)}$	$[27]_{(t)}+[29]_{(p,t)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$
Vertical mean	Vertical eddies
$1+[8]_{(\lambda)}+[27]_{(t)}+[34]_{(\lambda,t)}$	$[3]_{(p)}+[10]_{(p,\lambda)}+[29]_{(p,t)}+[36]_{(p,\lambda,t)}$
Longitude-time mean	Longitude-time eddies
$1+[3]_{(p)}$	$[8]_{(\lambda)}+[10]_{(p,\lambda)}+[27]_{(t)}+[29]_{(p,t)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$
Longitude-vertical mean	Longitude-vertical eddies
$1+[27]_{(t)}$	$[3]_{(p)}+[8]_{(\lambda)}+[10]_{(p,\lambda)}+[29]_{(p,t)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$
Vertical-time mean	Vertical-time eddies
$1+[8]_{(\lambda)}$	$[3]_{(p)}+[10]_{(p,\lambda)}+[27]_{(t)}+[29]_{(p,t)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$
Longitude-vertical-time mean	Longitude-vertical-time eddies
1	$[3]_{(p)}+[8]_{(\lambda)}+[10]_{(p,\lambda)}+[27]_{(t)}+[29]_{(p,t)}+[34]_{(\lambda,t)}+[36]_{(p,\lambda,t)}$

the complete space-time domain. The complexity of the standard mathematical notation may have been a deterrent factor in conducting such a three-dimensional study of atmospheric processes. It is hoped that the notation presented here will stimulate such "more-dimensional" investigations of the atmosphere's general circulation.

Equations (15) through (20), which contain additional terms, may have to be used for specific investigations which preclude averaging over all coordinates. For example, equation (20) will have to be applied if eddy transport terms are to be computed over individual longitude sectors $\lambda_i<\lambda$ (i indicating the sector number). Since $[(\ )_{(\lambda)}]_{(\lambda_i)}\neq 0$, the terms retained in equation (20), but not in equation (21), may be of significance. Similar considerations will hold for atmospheric layers $p_i<p$ and for time periods $\tau<t$. Equation (18) or (19), respectively, will have to be applied in such instances.

From the foregoing it appears that it may not be necessary to compute all 36 terms of equation (14), and the resulting terms in the various kinetic energy equations. No critique shall be voiced, therefore, of the many investigations carried out by numerous authors who have used approximate expressions. For the sake of completeness all terms, even those from which we may expect only negligible contributions, have been listed in the foregoing derivation. It would be worthwhile to investigate with actual data which of the terms in equations (14) through (21) may safely be neglected. Such computations should be carried out for both, the u- and v-components of motion, and for an extended period of time.

Once such an investigation has been made one may venture into the following problems:

1) In addition to the expressions for kinetic energy, those of momentum transport—involving terms u, v—and of potential energy should be explored in a similar fashion.

2) The energy conversion and transport equations, as for instance derived by Lorenz (1955) and Muench (1965), may be treated in a more detailed way than has been attempted so far.

3) The geographical latitude, ϕ, or sin ϕ, may be added as an additional domain. This, of course, would increase to 136 the number of terms in an equation similar to (14).

4) Each of the terms in the foregoing equations containing parentheses and yielding a worthwhile contribution, might be expanded in the wave number domain, as suggested by Saltzman (1957).

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